THE VOLUME OF THE MODULI SPACE OF FLAT CONNECTIONS ON A NONORIENTABLE 2-MANIFOLD

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ABSTRACT. We compute the Riemannian volume of the moduli space of flat connections on a nonorientable 2-manifold, for a natural class of metrics. We also show that Witten's volume formula for these moduli spaces may be derived using Haar measure, and we give a new proof of Witten's volume formula for the moduli space of flat connections on a 2-manifold using Haar measure.

1. INTRODUCTION

In [W] Witten defined and computed a volume on the moduli space of gauge equivalence classes of flat connections on a 2-manifold, using Reidemeister-Ray-Singer torsion (see e.g. [Fr]). When the 2-manifold is orientable, Witten proved that this volume is equal to the symplectic volume on the moduli space. However, when the 2-manifold is not orientable the moduli space does not have a symplectic form, so the interpretation of this volume has been unclear.

The moduli space of gauge equivalence classes of flat connections on a nonorientable 2-manifold turns out to be a Lagrangian submanifold of the moduli space of gauge equivalence classes of flat connections on the orientable double cover. In this article we compute its Riemannian volume for a natural class of metrics on the 2-manifold. The dependence on the choice of metric is discussed in Remarks 1 and 2 below.

The layout of this article is as follows. In Section 2 we compute the Riemannian volume of the moduli space of flat connections on a nonorientable 2-manifold (exhibited as the connected sum of a Riemann surface with either one or two copies of RP^2) using a metric derived from a particular class of metrics on $RP^2 - \{disc\}$. In Section 3 we give a new proof that Witten's formula for the volume of the moduli space of flat connections on 2-manifolds (for nonorientable surfaces with no boundary and orientable surfaces with one boundary component) arises from Haar measure.

2. Metrics on the moduli space

2.1. **Preliminaries.** By the classification of 2-manifolds [Ma], all nonorientable 2-manifolds are obtained as the connected sum of a Riemann surface with either one copy of the real projective plane RP^2 (denoted P) or two copies of P (which is equivalent to the connected sum with a Klein bottle K).

Given a Riemannian metric on the Möbius strip $P \setminus D$, we obtain a Riemannian metric on the connected sum $\Sigma \# P$ or $\Sigma \# P \# P$, where Σ is an oriented 2-manifold equipped with a Riemannian metric. We assume that a Riemannian metric has been chosen on $\Sigma \setminus D$.

We view $P \setminus D$ as formed by gluing together two sides of equal length of a triangle $\triangle$. We define a metric on $P \setminus D$ by endowing $\triangle$ with a Riemannian metric in which two sides are of equal length and using geodesic polar coordinates about one vertex of the triangle. In such coordinates (ρ, σ) (where ρ is the arc length from p and σ is the angular variable) the metric takes the form ([dC] Section 4.6)

$$
(1) \qquad\qquad ds^2 = d\rho^2 + g_{22}(\rho, \sigma) d\sigma^2
$$

We now assume that the triangle $\triangle$ is a geodesic triangle. In geodesic polar coordinates, the Hodge star operator is then

$$
(2) \qquad\qquad *d\rho = \sqrt{g_{22}} d\sigma
$$

$$
*d\sigma = -\frac{1}{\sqrt{g_{22}}} d\rho
$$

We make the assumption that $g_{22} = g_{22}(\rho)$ depends only on ρ and is independent of σ. This assumption is satisfied by the three important examples of metrics with constant scalar curvature:

(1) Spherical metric with scalar curvature $+1$: $g_{22}(\rho) = \sin^2(\rho)$
(2) Euclidean metric (with scalar curvature 0): $g_{22}(\rho) = \rho^2$
(3) Hyperbolic metric with scalar curvature -1: $g_{22}(\rho) = \sinh^2(\rho)$

We can consider two triangles shown in Figure 1. We use geodesic polar coordinates at the vertex x_0. The curves γ_1 and γ_2 are radial geodesics with constant σ. The curve γ_3 is the geodesic joining the two vertices x_1 and x_2. The curve C_1 is the curve $\rho = $ constant between x_1 and x_2; it is not a geodesic. The triangle $\triangle$ is bounded by $\gamma_1, \gamma_2, \gamma_3$;

it is a geodesic triangle. The triangle $\triangle_0$ is bounded by γ_1, γ_2, C_1. We can find the integral over the triangle $\triangle_0$ bounded by γ_1, γ_2, C_1 where C_1 is the curve specified by the equation $\rho = L = \text{constant}$.
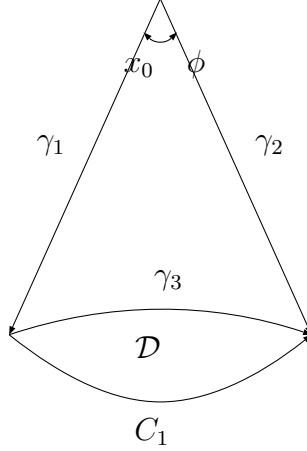

FIGURE 1. Triangles $\triangle$ and $\triangle_0$

Let G be a compact connected Lie group; the moduli space of flat connections on $\triangle$ modulo based gauge transformations (i.e. gauge transformations which take the value identity on the three vertices) can be identified with $G \times G$. The moduli space of flat connections on $P \setminus \mathcal{D}$ modulo gauge transformations which are trivial at one point on the boundary (corresponding to the vertices of the geodesic triangle) is equivalent to the subspace of the moduli space of flat connections on the geodesic triangle (modulo based gauge transformations) which correspond under an orientation reversing map which identifies two sides of the triangle . This is $\text{Hom}(\pi, G)$ where $\pi = \pi_1(P \setminus D)$, in other words

$$\text{Hom}(\pi, G) = \{(u, v) \in G \times G \mid uv^2 = 1\} \cong G.$$

Here v is the holonomy along γ_1 and $-\gamma_2$, while u is the holonomy along γ_3.

On the space of flat connections on $\triangle$, there is a metric

$$(3) \qquad <a, b> = \int_{\triangle} \text{Tr}(a \wedge *b)$$

where $*$ is the Hodge star operator on differential forms over $\triangle$. We will study the corresponding metric on $G \times G$. Define the metric $<,>$ at a flat connection A corresponding to a point p in $G \times G$ to be

$$< d_A \xi, d_A \eta > =: \int_{\triangle} \text{Tr}(d_A \xi \wedge *d_A \eta)$$

where $\xi(\cdot)$ and $\eta(\cdot)$ are $\mathfrak{g}$-valued functions on $\triangle$, so $d_A\xi$ and $d_A\eta$ represent elements of the tangent space to $G \times G$ at p. Here, $\mathrm{Tr}(\cdot)$ represents the ad-invariant inner product on the Lie algebra.

In the case when G is abelian, the holonomy of $d\xi$ along γ_1 is

$$(4) \quad Hol_{\gamma_1} d\xi = \exp \int_{\gamma_1} d\xi = \exp(\xi(\rho = L_1, \sigma = 0) - \xi(\rho = 0, \sigma = 0)),$$

and its holonomy along γ_2 is

$$(5) \quad Hol_{\gamma_2} d\xi = \exp \int_{\gamma_2} d\xi = \exp(\xi(\rho = L_2, \sigma = 0) - \xi(\rho = 0, \sigma = \phi)).$$

We will study these holonomies once we have determined ξ.

From now on, we only consider the inner product at $A = 0$. The reason is that the metric $< a, b >= \int \mathrm{Tr}(a \wedge *b)$ is ad-invariant (where $*$ denotes the Hodge star operator), and since the gauge action at the infinitesimal level is the adjoint action, this metric is invariant under the action of the gauge group and in particular under the action of the based gauge group. Since the space $\triangle$ is contractible, any flat connection is gauge equivalent to the trivial connection on a triangle. In other words all infinitesimal flat connections – elements of the tangent space of the space of flat connections modulo based gauge transformations – can be expressed as $d\xi$ for some infinitesimal gauge transformation $\xi : \triangle \to \mathfrak{g}$. So we need only consider the inner product at $A = 0$.

We want to associate a norm $< d\xi, d\xi >$ where $\xi \in \Omega^0(P \setminus D) \otimes \mathfrak{g}$. This will be done by defining

$$(6) \qquad\qquad < d\xi, d\xi >=: \int_{\triangle} \mathrm{Tr}(d\xi \wedge *d\xi)$$

Using Stokes' theorem, this can be converted to a line integral around the boundary of the triangle if we have $d * d\xi = 0$: this follows because

$$\int_{\triangle} \mathrm{Tr}(d\xi \wedge *d\xi) = \int_{\triangle} \mathrm{Tr}[d(\xi \wedge *d\xi) - \xi d * d\xi]$$

$$= \int_{\partial\triangle} \mathrm{Tr}\xi * d\xi - \int_{\triangle} \mathrm{Tr}\xi d * d\xi$$

In fact, we may assume ξ satisfies

$$(7) \qquad\qquad d * d\xi = 0.$$

This condition represents a transversal to the orbit of the based gauge group; our procedure is analogous to identifying de Rham cohomology classes with harmonic forms (forms α satisfying $d\alpha = 0$ and $d * \alpha = 0$). Note that the space $G \times G$ is isomorphic to the space of flat connections

on $\triangle$ modulo based gauge transformations (i.e. gauge transformations which take the value 1 at the vertices). Each equivalence class may be written as $d\xi$ for some $\xi : \triangle \to \mathfrak{g}$ (which does not take value 1 at all the vertices, unless one wishes to represent the trivial flat connection).

To solve equation (7), first, let us recall some geometry. Recall that our Riemannian metric in geodesic polar coordinates is

$$ds^2 = d\rho^2 + g_{22}(\rho)d\sigma^2$$

where ρ is the distance from a chosen point x_0 and σ is the polar angle with respect to this point. Let $*$ denote the Hodge star operator on the 2-manifold; the star operator with respect to these polar coordinates is

$$*d\rho = \sqrt{g_{22}(\rho)}d\sigma, \;\; *d\sigma = \frac{-1}{\sqrt{g_{22}(\rho)}}d\rho.$$

We use the substitution

(8)
$$u(\rho) = \int^{\rho} \frac{1}{\sqrt{g_{22}(t)}}dt$$

and define

$$\tau(\rho) = \exp u(\rho).$$

The equations for the Hodge star operator become

(9)
$$*du = d\sigma$$

$$*d\sigma = -du$$

We now have to solve the equation

$$d(*d\xi) = 0$$

We have

$$d\xi = \frac{\partial \xi}{\partial u}du + \frac{\partial \xi}{\partial \sigma}d\sigma$$

so

$$
\begin{aligned}
d(*d\xi) &= d(\frac{\partial \xi}{\partial u}d\sigma - \frac{\partial \xi}{\partial \sigma}du) = (\frac{\partial^2 \xi}{\partial \sigma^2} + \frac{\partial^2 \xi}{\partial u^2})du \wedge d\sigma \\
&= (\xi_{\sigma\sigma} + \xi_{uu})du \wedge d\sigma
\end{aligned}
$$

So the equation we need to solve is

(10)
$$\xi_{\sigma\sigma} + \xi_{uu} = 0.$$

Let us take a Fourier decomposition of ξ in the σ variable:

$$\xi(u, \sigma) = \int_{\mathbb{R}} \hat{\xi}(u, k)e^{ik\sigma}dk$$

So (10) becomes

$$(11) \qquad \frac{\partial^2}{\partial u^2} \hat{\xi}(u, k) - k^2 \hat{\xi}(u, k) = 0$$

This equation (11) has the following solutions:

$$\hat{\xi}(u, k) = C_+(k)\tau^k + C_-(k)\tau^{-k}.$$

Recall that we had defined $\tau = \exp(u)$. Imposing the condition that $\hat{\xi}(u, k)$ is finite at $u = 0$, we get

$$\hat{\xi}(u, k) = C_+(k)\tau^k$$

when $k > 0$, and

$$\hat{\xi}(u, k) = C_-(k)\tau^{|k|}$$

when $k < 0$. Recall that ϕ is the polar angle of the geodesic triangle. Let us impose the additional constraint that $\xi(u, \sigma)$ attains its maximum on two edges of the triangle (γ_1 which is $\sigma = 0$ and γ_2 which is $\sigma = \phi$), in other words $\frac{\partial}{\partial \sigma}\xi(\rho, \sigma) = 0$ when $\sigma = 0$ and $\sigma = \phi$. This leads to

$$\xi(u, \sigma) = Y\tau^k \cos k\sigma$$

where $Y \in \mathfrak{g}$ is a constant and

$$(12) \qquad k = \pi/\phi.$$

We compute that

$$(13) \qquad \int_{C_1} \mathrm{Tr}(\xi \wedge *d\xi) = \int_0^\phi < \xi, \partial\xi/\partial u > d\sigma$$

$$=< Y, Y > k \exp ku(L) \int_0^\phi \cos^2 k\sigma d\sigma$$

$$=< Y, Y > \pi/2 \exp ku(L).$$

Note that writing ξ in terms of its Fourier decomposition accomplishes the same thing as solving the equation $d*d\xi = 0$ by the method of separation of variables. We solved this equation in the previous subsection. Thus we obtain

$$(14) \qquad \xi(\rho, \sigma) = Y\tau(\rho)^k \cos k\sigma, \ \ k = \pi/\phi$$

so that the maximum of $|\xi|$ is achieved on the edges $\sigma = 0$ and $\sigma = \phi$ and $Y \in \mathfrak{g}$ is a variable we choose so that we can get the desired

holonomy along the boundary. The holonomies were specified by the equations (4) and (5).

$$(15) \qquad \int_{C_1} \mathrm{Tr}(\xi \wedge *d\xi) = < Y, Y > k \left[\tau(\frac{L}{2})\right]^{2k} \int_0^\phi d\sigma \cos^2 k\sigma$$

$$= < Y, Y > \left[\tau(\frac{L}{2})\right]^{2k} \frac{\pi}{2}$$

$$\int_{\gamma_1} \mathrm{Tr}(\xi \wedge *d\xi) = \int_{\gamma_2} \mathrm{Tr}(\xi \wedge *d\xi) = 0$$

Thus (referring to Figure 1) the integral over the geodesic triangle $\triangle$ bounded by $\gamma_1, \gamma_2, \gamma_3$ is the integral over the triangle $\triangle_0$ bounded by γ_1, γ_2, C_1 minus the integral over the region $\mathcal{D}$:

$$(16) \qquad \int_\triangle \mathrm{Tr}(d\xi \wedge *d\xi) = \int_{\triangle_0} \mathrm{Tr}(d\xi \wedge *d\xi) - \int_{\mathcal{D}} \mathrm{Tr}(d\xi \wedge *d\xi).$$

This leads to

$$(17) \qquad \int_\triangle \mathrm{Tr}(d\xi \wedge *d\xi) = \int_{C_1} \mathrm{Tr}(\xi \wedge *d\xi) - \int_{\mathcal{D}} \mathrm{Tr}(d\xi \wedge *d\xi).$$

The integral over $\mathcal{D}$ is given as follows:

$$\int_{\mathcal{D}} \mathrm{Tr}(d\xi \wedge *d\xi) = \int_{\mathcal{D}} < Y, Y > k^2 \tau^{2k} (\frac{\cos^2 k\sigma}{\tau} + \frac{\sin^2 k\sigma}{\tau}) d\tau \wedge d\sigma$$

$$= \int_{\mathcal{D}} < Y, Y > k^2 \tau^{2k} \frac{1}{\tau} d\tau \wedge d\sigma$$

$$= < Y, Y > k^2 \int_0^\phi d\sigma \int_{\tau(\gamma_3(\sigma))}^{\tau(C_1(\sigma))} \tau^{2k} \frac{d\tau}{\tau}$$

$$(18) \qquad = < Y, Y > k^2 \int_0^\phi (\frac{\tau^{2k}(C_1(\sigma))}{2k} - \frac{\tau^{2k}(\gamma_3(\sigma))}{2k}) d\sigma$$

where $\tau(C_1(\sigma))$ is a constant independent of σ.

Remark 1. *If g is a Riemannian metric on $\triangle$ then the quantity*

$$< d\xi, d\xi >_g$$

(associated to the metric g) is equal to the quantity $< d\xi, d\xi >_{ge^w}$ where $w : P \setminus D \to \mathbb{R}$ is a C^∞ function. In other words our definition of the metric $< d\xi, d\xi >_g$ is invariant under conformal transformations on $\triangle$. To see this, we write the metric as

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

with inverse

$$g^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}.$$

We observe that the Hodge star operator is unchanged under $g \mapsto ge^w$ (where $w : \triangle \to \mathbb{R}$), since $\det g$ transforms to $(\det g)e^{2w}$ while g^{11} transforms to $g^{11}e^{-w}$. The Hodge star operator for coordinates x_1, x_2 is

$$*dx_1 = g^{11}\sqrt{\det g}dx_2$$

Hence $g^{11}\sqrt{\det g}$ is unchanged. Thus the norm (denoted by Γ) on $d\xi$ computed using metrics of the form

(19) $$ds^2 = d\rho^2 + g_{22}(\rho)d\sigma^2$$

also gives the norm for metrics g conformally equivalent to those of the form (19).

Remark 2. *In fact in dimension 2 every Riemannian metric is locally diffeomorphic to one which is conformally equivalent to a metric of constant curvature (see [d'H] Section 3.3). This shows that all metrics on $P \setminus D$ are conformally equivalent to a metric of constant curvature, one of the three listed in (2.1), for which the norm (6) may be computed as in (16). The values of the norms are different for the three different choices of constant curvature metrics.*

Remark 3. *The volume of the moduli space defined using Reidemeister-Ray-Singer torsion is independent of the choice of metric on the 2-manifold either orientable or nonorientable. The Riemannian volume, however, is a different story. For an orientable 2-manifold, the moduli space of flat connections is a Kähler manifold, so the Riemannian volume equals the symplectic volume, and is thus independent of the choice of metric. For a nonorientable surface, Witten remarks in [W] (p. 163) that the Riemannian volume does depends on the metric. According to [W] (2.38) the torsion volume and the Riemannian volume differ (in this case) by a ratio of determinants of elliptic operators (generalized Laplacians on Lie algebra-valued differential forms). Our results are consistent with this observation, and at the same time provide an evaluation of this ratio of determinants.*

2.2. **Hyperbolic metric on the geodesic triangle.** We explore the hyperbolic case first. Note that if Σ has genus ≥ 2 then by the uniformization theorem it is equipped with a unique metric of constant scalar curvature -1 (in other words a hyperbolic metric). This is one motivation for choosing a hyperbolic metric on $P \setminus D$, although such a metric will be singular at one point.

A Riemann surface with one boundary component and genus ≥ 1 always has a hyperbolic metric with constant curvature -1 [B]. For this reason we construct a metric on $P \setminus D$ for which the boundary is a geodesic and which is hyperbolic with constant curvature -1 at all but one point. We do this by gluing the edges of an isosceles geodesic hyperbolic triangle using an orientation reversing map: this process identifies all three vertices of the triangle, and the vertex becomes the point where the metric does not have constant curvature -1 (indeed, the metric is singular at this point).

Remark 4. *The reason why this procedure gives a natural metric on the 2-manifold formed by taking the connected sum of a Riemann surface of genus $\ell > 0$ with P is that a Riemann surface of genus $\ell > 0$ with one or two boundary components always has a hyperbolic metric with constant curvature for which the boundary components are geodesics. This metric can be obtained from a pants decomposition of the Riemann surface, where each pair of pants is equipped with a hyperbolic metric for which the boundary components are geodesics. See [B].*

Hence the connected sum of a Riemann surface with P has a metric which is hyperbolic at all but one point, and the connected sum of a Riemann surface with two copies of P has a metric which is hyperbolic at all but 2 points. To form P with one disc removed, we recall that the two edges of the triangle are identified using an orientation reversing map. In Figure 1 this corresponds to identifying γ_1 with γ_2 using an orientation reversing map which maps $x_1 \in \gamma_1$ to $x_0 \in \gamma_2$ and maps $x_0 \in \gamma_1$ to $x_2 \in \gamma_2$. It turns out that the metric on the based moduli space on P with one disc removed is singular at the vertex.
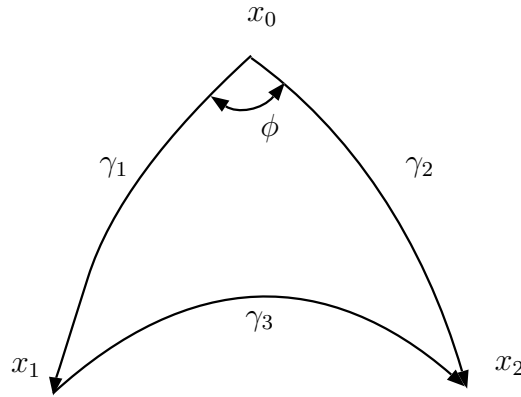

FIGURE 2. Hyperbolic geodesic triangle

We have introduced a geodesic triangle $\triangle$ in the upper half plane (see Figure 2). As shown in Figure 2, the geodesic triangle has one distinguished vertex x_0 (which serves as the origin for geodesic polar coordinates). The sides $\gamma_1, \gamma_2, \gamma_3$ are geodesics. We introduce geodesic polar coordinates (ρ, σ) (using the hyperbolic metric on the upper half plane, which is assumed to contain the triangle) with x_0 as the origin: here ρ is the distance from a chosen point x_0 and σ is the polar angle. It is assumed that the geodesic γ_1 is defined by the equation $\sigma = 0$, and γ_2 is defined by $\sigma = \phi$ (where ϕ is the angle at x_0). The curves γ_1 and γ_2 have lengths L_1 and L_2 which are the same here. The geodesic γ_3 is specified by Coxeter's equation

$$(20) \qquad \coth \rho = \frac{\cos(\sigma - \sigma_0)}{\ell}, \quad \text{see [Cox] p.376,}$$

where ℓ, σ_0 are constants determined by the geodesic γ_3. Also,

$$\coth \rho = \frac{\cosh \rho}{\sinh \rho} = \frac{\cosh^2(\rho/2) + \sinh^2(\rho/2)}{2\sinh(\rho/2)\cosh(\rho/2)} = \frac{1}{2}(\tau + \frac{1}{\tau})$$

for $\tau = \tanh(\rho/2)$. Thus

$$(1 - \frac{\cos(\sigma - \sigma_0)}{\ell \tau})d\tau = \frac{-\sin(\sigma - \sigma_0)}{\ell}d\sigma.$$

By (20), it is clear that since $\rho(\sigma = 0)$ must be equal to $\rho(\sigma = \phi)$ for a hyperbolic geodesic triangle two sides of which will be identified to form $P \setminus D$, we need $\sigma_0 = \phi/2$ in (20).

Now the equation for γ_3 is (see equation (20))

$$\cos(\sigma - \phi/2)\tanh \rho = \cos(\phi/2)\tanh L$$

and

$$\tanh \rho = \frac{2\tau}{1 + \tau^2}.$$

So we have
(21)
$$\tau(\gamma_3(\sigma)) = \frac{\cos(\sigma - \phi/2)}{l} - \sqrt{(\frac{\cos(\sigma - \phi/2)}{l})^2 - 1}, \ l = \cos(\phi/2)\tanh L$$

Thus by (18)

$$\int_{\mathcal{D}} \text{Tr}(d\xi \wedge *d\xi) = \frac{k\phi}{2} <Y,Y> \left[\tanh(\frac{L}{2})\right]^{2k}$$

$$-\frac{k}{2} <Y,Y> \int_0^\phi \left[\frac{\cos(\sigma - \phi/2)}{l} - \sqrt{(\frac{\cos(\sigma - \phi/2)}{l})^2 - 1}\right]^{2k} d\sigma.$$

Let $x = \cos(\sigma - \phi/2)$;

$$(22) \qquad \int_{\mathcal{D}} \mathrm{Tr}(d\xi \wedge *d\xi) \;\; = \;\; \frac{k\phi}{2} < Y, Y > \left[\tanh(L/2)\right]^{2k}$$

$$-\frac{k}{2} < Y, Y > \left[\int_{\cos(\phi/2)}^{1} \left(\frac{x}{l} - \sqrt{(\frac{x}{l})^2 - 1}\ \right)^{2k} \frac{dx}{\sqrt{1-x^2}}\right.$$

$$\left. -\int_{1}^{\cos(\phi/2)} \left(\frac{x}{l} - \sqrt{(\frac{x}{l})^2 - 1}\ \right)^{2k} \frac{dx}{\sqrt{1-x^2}}\right]$$

$$=< Y, Y > \frac{k\phi}{2}\left[\tanh(L/2)\right]^{2k} - k < Y, Y > \int_{\cos(\phi/2)}^{1} \left(\frac{x}{l} - \sqrt{(\frac{x}{l})^2 - 1}\ \right)^{2k} \frac{dx}{\sqrt{1-x^2}}$$

Consider a more sp ecific geodesic triangle, the triangle with two sides of equal length L. This means σ is fixed on those two sides and the third side is determined by Coxeter's equation (20), where (ρ, σ) are the polar coordinates as defined above. By (14) we have

$$\xi(\rho, \sigma) = Y(\exp u(\rho))^k \cos k\sigma$$

where in the notation of (8), $u = \tanh(\rho/2)$ in the case of a hyperbolic metric.

We recall from (17) that

$$\int_{\triangle} \mathrm{Tr}(d\xi \wedge *d\xi) = \int_{C_1} \mathrm{Tr}(\xi \wedge *d\xi) - \int_{\mathcal{D}} \mathrm{Tr}(d\xi \wedge *d\xi)$$

$$(23) \qquad = k < Y, Y > \int_{\cos(\phi/2)}^{1} (\frac{x}{l} - \sqrt{(\frac{x}{l})^2 - 1}\ )^{2k} \frac{dx}{\sqrt{1-x^2}}$$

We will compute this integral in Appendix A.

We denote the integral (23) by $k < Y, Y > h(\phi)$ because although it depends on two parameters ϕ and L, there is a relation between these two. We can see this as follows. Assume the length of γ_3 is fixed (since γ_3 will be glued to the boundary of a Riemann surface); denote this length by b. We have

$$\cosh^2 L = \frac{\sinh^2 b}{2(\cosh b - 1)} + \frac{(\cosh b - 1)}{2}\frac{(1 + \cos \phi)^2}{1 - \cos^2 \phi}$$

$$\frac{2(\cosh b - \cos \phi)(\cosh b - 1)(\cos \phi + 1)}{2(\cosh b - 1)(1 - \cos^2 \phi)}$$

$$(24) \qquad\qquad = \frac{\cosh b - \cos \phi}{1 - \cos \phi}$$

This is the relation between ϕ and L once b is fixed.

If $\tau(L) = \tanh L/2$, then the relation between $\tau(L)$ and ϕ is

$$(25) \qquad \tau^2(\phi) = \tau^2(L(\phi)) = \frac{\sqrt{\frac{\cosh b - \cos \phi}{1 - \cos \phi}} - 1}{\sqrt{\frac{\cosh b - \cos \phi}{1 - \cos \phi}} + 1};$$

thus h is a function of the top angle ϕ only.

Thus from (23) we have

$$(26) \qquad < d\xi, d\xi > = k < Y, Y > h(\phi)$$

If we choose an orthonormal basis $e_1, e_2, \cdots, e_n$ for $Lie(G)$, then the volume element of this moduli space using the metric Γ is $\sqrt{\det \Gamma}$, and

$$\det \Gamma = (kh(\phi))^n, \text{ using } (26)$$

where n is the dimension of $Lie(G)$ i.e. $\mathfrak{g}$.

2.3. Euclidean metric on the geodesic triangle.

The only smooth constant curvature metric on $P \backslash D$ for which the boundary is a geodesic is the Euclidean metric. For the geodesic triangle $\triangle$ equipped with the Euclidean metric, we get

$$u = \int^\rho \frac{d\rho}{\rho} = \ln \rho$$

so $\tau = \rho$ and

$$\xi = Y \rho^k \cos k\sigma.$$

The geodesic γ_3 is given by

$$\rho \cos(\sigma - \phi/2) = L$$

so we have

$$(27) \qquad \int_{C_1} \text{Tr}(\xi \wedge *d\xi) = \frac{\pi}{2} < Y, Y > L^{2k}$$

and from (16) and (18)

$$\int_\mathcal{D} \text{Tr}(d\xi \wedge *d\xi) = \frac{\pi}{2} < Y, Y > L^{2k} - \frac{k}{2} < Y, Y > \int_{\rho \cos(\sigma - \phi/2) = L} \rho^{2k} d\sigma$$

$$(28) \qquad = \frac{\pi}{2} < Y, Y > L^{2k} - k < Y, Y > L^{2k} \int_0^{\phi/2} \sec^{2k}(\sigma) d\sigma.$$

Thus the norm $\int_\mathcal{D} \text{Tr}(d\xi \wedge *d\xi)$ is given by $k < Y, Y > h(\phi)$ for an appropriate function $h(\phi)$ which depends on the angle ϕ and on the choice of Euclidean metric.

2.4. **Spherical metric on the geodesic triangle.** For a spherical metric on the geodesic triangle, we get

$$(29) \qquad u = \int^{\rho} \frac{dx}{\sin x} = \ln \tan(\rho/2)$$

so $\tau = \tan(\rho/2)$. We then get that

$$(30) \qquad \int_{C_1} \mathrm{Tr}(\xi \wedge *d\xi) = \frac{\pi}{2} < Y, Y > (\tan(L/2))^{2k}$$

Hence we get that
(31)

$$\int_{\mathcal{D}} \mathrm{Tr}(d\xi \wedge *d\xi) = < Y, Y > \left\{ \frac{\pi}{2}(\tan L/2)^{2k} - \frac{k}{2} \int_{\gamma_3} (\tan \rho/2)^{2k} d\sigma \right\}.$$

Here, γ_3 is the segment of the great circle connecting the points $x_1 = (\rho = L, \sigma = 0)$ and $x_2 = (\rho = L, \sigma = \phi)$. The length of this geodesic and the angles at the vertices x_1, x_2 can be determined using spherical trigonometry [Weis]. Thus when $\triangle$ is equipped with a spherical metric of constant curvature, the norm $< d\xi, d\xi >$ is given by $k < Y, Y > h(\phi)$ for an appropriate function $h(\phi)$ which depends on ϕ and on the choice of spherical metric.

2.5. **Σ is the connected sum of a Riemann surface with P.** To consider the volume of the moduli space of the connected sum of a Riemann surface with P, first, let us examine the following lemmas:

Lemma 5. *Let E be a Riemannian manifold with metric Γ. Let $f : E \to \mathbb{R}$ be a smooth function for which $df_m \neq 0$ for any $m \in E$. Then*

$$Vol(E) = \int_{\mathbb{R}} \frac{Vol f^{-1}(t)}{df(v(t))} dt$$

where $v(t)$ is the unit normal vector to $f^{-1}(t)$.

Proof. The volume form on $T_m E$ is $e_1^* \wedge \cdots \wedge e_n^*$ where $\{e_j\}$ is an orthonormal basis of tangent vectors i.e. $\Gamma(e_j, e_k) = \delta_{jk}$ and e_i^* are the dual basis vectors for $T_m^* E$ for any $m \in E$. Choose $e_1, \cdots, e_{n-1} \in T_m(f^{-1}(t))$ so $e_1^* \wedge \cdots \wedge e_{n-1}^*$ is the volume form on $T_m(f^{-1}(t))$. Then

$e_n^* = \frac{df}{df(v(t))}$ since $v(t)$ is the unit vector normal to $f^{-1}(t)$. Thus

$$
\begin{aligned}
Vol(E) &= \int_E (e_1^* \wedge \cdots \wedge e_{n-1}^*) \wedge e_n^* \\
&= \int_E (e_1^* \wedge \cdots \wedge e_{n-1}^*) \wedge \frac{df}{df(v(t))} \\
&= \int_{t \in \mathbb{R}} dt \int_{f^{-1}(t)} \frac{(dvol)}{df(v(t))} \\
&= \int_{\mathbb{R}} \frac{Vol(f^{-1}(t))}{df(v(t))} dt
\end{aligned}
$$

$\square$

Similarly, we have the following lemma,

Lemma 6. *Let E be a Riemannian manifold with metric Γ, and suppose $f : E \to G$ is a smooth map for which df has maximal rank at generic points in E. Then*

$$
Vol(E) = \int_{g \in G} \frac{Vol(f^{-1}(g))}{dg(\wedge_{j=1}^N df(v_j))} dg
$$

where dg is the volume element given by a Riemannian metric on G and $v_1, \cdots, v_N$ are unit normal vectors to $f^{-1}(g)$ with $N = dim\ G$.

Now let us return to our particular situation.[1] We have a nonorientable surface Σ which is the connected sum of a Riemann surface Σ_1 with a nonorientable surface Σ_2 (either one or two copies of the projective plane P). Denote $E_1 = Hom(\pi_1(\Sigma_1), G)$ where Σ_1 is a Riemann surface of genus ℓ with one boundary component, and $E_2 = Hom(\pi_1(\Sigma_2), G)$ where in this section $\Sigma_2 = P$ with one disc removed. Here $\pi_1(\Sigma_2) = \{x, y | x = y^2\} \cong Z$ so $Hom(\pi_1(\Sigma_2), G) \cong G$. Define maps $f_i : E_i \to G$ for $i = 1, 2$ by sending a representation to its value on the loop around the boundary. Let

$$
E = \{(m_1, m_2) \in E_1 \times E_2 \mid f_1(m_1) = f_2(m_2)\}
$$

Now we use Lemma 6 with $E_2 \cong G$ and let $f : E \to G = E_2$ be the map $(m_1, m_2) \mapsto m_2$. Notice that in this case the hypothesis of f having maximal rank is valid generically because this reduces to $df_1 : TE_1 \to \mathfrak{g}$ being surjective, and it was proved by Goldman [Go1] (Proposition 3.7) that the image of df_1 at a point $A = (a_1, \ldots, a_\ell, b_1, \ldots, b_\ell) \in E_1 = G^{2\ell}$ is $z(A)^\perp$ (the orthocomplement of the Lie algebra of the stabilizer of A under the adjoint action), and $z(A) = 0$ for generic A.

---

[1]For other approaches to measures on moduli spaces, see [Fine], [Fo], [Liu] and [BeSe, Se1, Se2, Se3].

We have

$$Vol(E) = \int_G \frac{Vol(f^{-1}(g))}{dg(\wedge_j df(v_j))} dg.$$

Note that the metric on E is $\pi_1^* h_1 + \pi_2^* h_2$ where h_1 is the metric on E_1 and h_2 the metric on E_2 which are given by the equation (26) i.e. (Haar measure) $\times (kh(\phi))^{n/2}$. Thus we can choose an orthonormal basis $\{v_j\}$ in $\mathfrak{g}$ and $dg(\wedge_j df(v_j)) = 1$. Thus we have $Vol(f^{-1}(g)) = Vol(f_1^{-1}(g^2))Vol(f_2^{-1}(g^2))$ because $f^{-1}(g) = f_1^{-1}(g^2) \times f_2^{-1}(g^2)$. It follows that

$$Vol(E) = \int_{g \in G} Vol(f_1^{-1}(g^2))Vol(f_2^{-1}(g^2)) dg$$

where our moduli space is $\mathcal{M} = E/G$. Thus we have $Vol(E) = Vol(\mathcal{M})Vol(G)$.

Notice that the moduli space of gauge equivalence classes of flat connections on a Riemann surface with one boundary component about which the holonomy is constrained to take a fixed value (in other words the moduli space of parabolic bundles) is a Kähler manifold (see for instance [AB]) so its symplectic volume (as specified by Witten's formula) is equal to its Riemannian volume (for a metric derived from any metric on the Riemann surface). Hence our procedure will give the Riemannian volume on the moduli space of gauge equivalence classes of flat connections on the connected sum.

We know from [W] the volume formula [W](4.114) for a moduli space of flat connections on a compact orientable surface Σ_ℓ of genus ℓ with one boundary component:[2] For $s \in G$, we define
(32)

$$\mathcal{M}(\Sigma_\ell, s) = \{(a_1, \ldots, a_{2\ell}) \in G^{2\ell} \mid \prod_{j=1}^{\ell} a_{2j-1}a_{2j}a_{2j-1}^{-1}a_{2j}^{-1} \in C(s)\}/G$$

$$= \{\rho \in \mathrm{Hom}(\pi_1(\Sigma_\ell - D), G) \mid \rho([\partial(\Sigma_\ell - D)]) \in C(s)\}/G$$

where $C(s)$ is the conjugacy class of s. We also define

(33) $$\mathcal{R}(\Sigma_\ell, s) = \{(a_1, \ldots, a_{2\ell}) \in G^{2\ell} \mid \prod_{j=1}^{\ell} a_{2j-1}a_{2j}a_{2j-1}^{-1}a_{2j}^{-1} = s\}$$

$$= \{\rho \in \mathrm{Hom}(\pi_1(\Sigma_\ell - D), G) \mid \rho([\partial(\Sigma_\ell - D)]) = s\}$$

so that

$$\mathcal{M}(\Sigma_\ell, s) = G\mathcal{R}(\Sigma_\ell, s)/G.$$

---

[2]In Section 3.1 we give a new proof of this formula.

It follows that

$$(34) \qquad Vol\mathcal{M}(\Sigma_\ell, s) = Vol\mathcal{R}(\Sigma_\ell, s)\frac{VolC(s)}{VolG}$$

If $s \in G$, then Witten's formula reads

$$(35) \qquad Vol(\mathcal{M}(\Sigma_\ell, s)) = C_1 \sum_\alpha \frac{1}{(\dim\alpha)^{2\ell-1}}\chi_\alpha(s)\sqrt{F(s)},$$

where α runs over all isomorphism classes of irreducible representations of G, the representation α has character χ_α, and the constant C_1 is

$$(36) \qquad C_1 = \frac{\sharp Z(G)Vol(G)^{2\ell-2+1}}{(2\pi)^{\dim\mathcal{M}(\Sigma_\ell,s)}Vol(T)}$$

and

$$\dim\mathcal{M}(\Sigma_\ell, s) = (2\ell - 2) \mid G \mid + \mid G \mid - \mid T \mid,$$

where $|G|$ denotes the dimension of G and $|T|$ denotes the dimension of the maximal torus T. Note also that $F(s)$ is the Riemannian volume of the conjugacy class $C(s)$ through s as defined in [W](4.53)

$$v = \frac{Vol(G)}{Vol(T)}v_0 F(s).$$

(See Chapter 7, [BGV].) The Liouville volume of $C(s)$ is

$$(37) \qquad \sqrt{F(s)}\text{vol}G/\text{vol}T$$

(see [AMW], Proposition 3.6.) Here v represents the measure on T/W obtained by pushing down the Haar measure on G (under the natural map from a group element to its conjugacy class) and v_0 represents the measure on T/W obtained by restricting the metric on $\mathfrak{g}$ to $\text{Lie}(T)$ and then identifying $\text{Lie}(T)$ with the tangent space to T/W. A more detailed explanation of $F(s)$ is given in the next section.

So (34) is equivalent to

$$(38) \qquad Vol\mathcal{M}(\Sigma_\ell, s) = Vol\mathcal{R}(\Sigma_\ell, s)\frac{\sqrt{F(s)}}{VolT},$$

and (35) is equivalent to

$$(39) \qquad Vol\mathcal{R}(\Sigma_\ell, s) = C_1 VolT \sum_\alpha \frac{1}{(\dim\alpha)^{2\ell-1}}\chi_\alpha(s).$$

Thus the volume of the moduli space of flat connections on the connected sum of a Riemann surface with a projective plane is

$$(40) \qquad Vol(\mathcal{R}(\Sigma)) = Vol(\mathcal{R}(\Sigma_\ell\sharp P))$$

$$= \int_{s \in G} Vol(\mathcal{R}(\Sigma_\ell, s)) Vol(\mathcal{R}(P, s)) \, ds$$

Denote by R the map $G \to G$,

(41)
$$R : g \mapsto g^2.$$

Weyl's integral formula

$$|W| \int_G f(g) dg = \int_T [F(t) \int_G f(gtg^{-1}) dg] dt$$

gives us that

$$\int_G f(g) dg = \int_{T/W} F(s) f(s) \frac{Vol(G)}{Vol(T)} ds.$$

when f is conjugation invariant.

Thus

$$\begin{aligned} Vol(\mathcal{R}(\Sigma)) &= \int_G C_1 Vol T \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \chi_\alpha(g) R_*(\sqrt{\det\Gamma} dg) \\ &= C_1 Vol(T) \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \int_G \chi_\alpha(g^2) \sqrt{\det\Gamma} dg \end{aligned}$$

The above is true because $S^2 - D - D$ is the double cover of $P \setminus D$; the space of flat connections on $S^2 - D - D$ (modulo based gauge transformations) is isomorphic to G, so we pull back the integral to G under the covering map. We get

$$Vol(\mathcal{R}(\Sigma)) = C_1 Vol T \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} f_\alpha Vol(G)(kh(\phi))^{|G|/2}$$

by using

(42)
$$\int_G \chi_\alpha(g^2) dg = f_\alpha Vol(G)$$

(cf:[W] (2.70)) where $f_\alpha = 1, -1, 0$ depending on whether the representation α admits a symmetric invariant bilinear form, an antisymmetric bilinear form, or no invariant bilinear form at all. Here $\dim(\alpha)$ is the dimension of the representation α, and its character is denoted by χ_α.

We get

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} H(\phi) \sum_\alpha \frac{f_\alpha}{(\dim(\alpha))^{2\ell-1}}$$

where $H(\phi) = (kh(\phi))^{|G|/2}$ and $k = \pi/\phi$. Here h is a real-valued function of the angle ϕ which depends on the choice of metric on the Möbius strip.

We compare our result to Witten's formula [W](4.93)

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|}} \sum_\alpha \frac{f_\alpha}{(\dim(\alpha))^{2\ell-1}}$$

Our formula differs from Witten's by a multiplicative factor of $H(\phi)$ $(2\pi)^{|T|}$. The factor $(2\pi)^{|T|}$ is due to Witten's choice of a different normalization. The factor $H(\phi)$ results from our choice of a metric on the nonorientable part of the surface (the projective plane). Note Witten did not choose a metric on P. On the other hand our formula for $Vol(\mathcal{M})$ is a function of the angle ϕ for a fixed surface.

2.6. Σ is a connected sum of a Riemann surface with two projective planes. We know from [W] the volume formula for a moduli space of flat connections on a compact orientable surface of genus ℓ with two boundary components [W] (4.114) (where $s_1, s_2 \in G$ and the holonomies around the two boundary components are fixed at s_1 and s_2): in this case Witten's formula [W] (4.114) reads as follows.
(43)
$$Vol(\mathcal{M}(\Sigma_\ell, s_1, s_2)) = C_2 \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell}} \chi_\alpha(s_1) \sqrt{F(s_1)} \chi_\alpha(s_2) \sqrt{F(s_2)},$$

where the constant C_2 is

$$\frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{\dim \mathcal{M}(\Sigma_\ell, s_1, s_2)} Vol(T)^2},$$

and

$$\dim \mathcal{M}(\Sigma_\ell, s_1, s_2) = (2\ell - 2) \mid G \mid +2 \mid G \mid -2 \mid T \mid .$$

Again we define

$$\mathcal{R}(\Sigma_\ell, s_1, s_2) = \{\rho \in \text{Hom}(\pi_1(\Sigma_\ell - D_1 - D_2), G) \mid \rho([\partial D_1] = s_1, \ \rho([\partial D_2] = s_2\}.$$

Here we have chosen representatives $[\partial D_j]$ for the elements of the fundamental group represented by the loops around the j-th boundary component, by connecting the basepoint to some point on the boundary. Thus

$$\mathcal{M}(\Sigma_\ell, s_1, s_2) = (G \times G)\mathcal{R}(\Sigma_\ell, s_1, s_2)/G \times G$$

where $(g_1, g_2) \in G \times G$ acts on $\text{Hom}(\pi_1(\Sigma_\ell - D_1 - D_2), G)$ so that if the value of the representation of the loop around the j-th boundary component is s_j, then this value becomes $g_j s_j g_j^{-1}$. Following (34) and (37) above, we have

(44) $$Vol\mathcal{M}(\Sigma_\ell, s_1, s_2) = Vol\mathcal{R}(\Sigma_\ell, s_1, s_2)\frac{VolC(s_1)VolC(s_2)}{(VolG)^2}$$

$$= Vol\mathcal{R}(\Sigma_\ell, s_1, s_2)\frac{\sqrt{F(s_1)}\sqrt{F(s_2)}}{(VolT)^2}.$$

Equivalently, Witten's formula is

$$Vol(\mathcal{R}(\Sigma_\ell, s_1, s_2)) = C_2(Vol(T))^2 \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell}}\chi_\alpha(s_1)\chi_\alpha(s_2)$$

Thus the volume of the moduli space of flat connections on the connected sum of a Riemann surface with two projective planes (equivalently, the connected sum with one Klein bottle) is

$$Vol(\mathcal{R}(\Sigma)) = Vol(\mathcal{R}(\Sigma_\ell \sharp P \sharp P))$$
$$= \int_{G\times G} ds_1 ds_2 Vol(\mathcal{R}(\Sigma_\ell, s_1, s_2))Vol(\mathcal{R}(P, s_1))Vol(\mathcal{R}(P, s_2))$$

Let $\sqrt{\det\Gamma_1}$ and $\sqrt{\det\Gamma_2}$ denote the volume elements on the two copies of $P \setminus D$ respectively. We have
(45)
$$Vol(\mathcal{R}(\Sigma)) = \int_{G\times G} Vol(\mathcal{R}(\Sigma_\ell, g_1, g_2))R_*(\sqrt{\det\Gamma_1}dg_1)R_*(\sqrt{\det\Gamma_2}dg_2)$$

$$= C_2(VolT)^2 \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell}}\int_{G\times G}\chi_\alpha(g_1)\chi_\alpha(g_2)R_*(\sqrt{\det\Gamma_1}dg_1)R_*(\sqrt{\det\Gamma_2}dg_2)$$

$$= C_2(VolT)^2 \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell}}\sqrt{\det\Gamma_1}\sqrt{\det\Gamma_2}(\int_G\chi_\alpha(g^2)dg)^2$$

where R was defined by (41). We use equation (42) and thus the volume becomes

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G)Vol(G)^{2\ell-1}}{(2\pi)^{(2\ell)|G|-2|T|}} \sum_\alpha \frac{1}{(\dim(\alpha))^{2l}}f_\alpha^2 H(\phi_1)H(\phi_2)Vol(G)^2$$

where $H(\phi_i) = (k_i h(\phi_i))^{|G|/2}$, the angles ϕ_1 and ϕ_2 are the top angles of the two triangles respectively, $k_i = \pi/\phi_i$, and f_α is defined as in Section 2.5. Note that $f_\alpha^2 = 1$ if $\alpha = \bar{\alpha}$ and $f_\alpha^2 = 0$ otherwise. We get the final formula

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G)Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell)|G|-2|T|}}H(\phi_1)H(\phi_2)\sum_{\alpha=\bar{\alpha}}\frac{1}{(\dim(\alpha))^{2\ell}}.$$

We compare this with Witten's formula [W](4.77)

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G)Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell)|G|}}\sum_{\alpha=\bar{\alpha}}\frac{1}{(\dim(\alpha))^{2\ell}}$$

Our formula differs from Witten's formula by a multiplicative factor of $H(\phi_1)\, H(\phi_2)\, (2\pi)^{2(|T|)}$. The factor $(2\pi)^{2(|T|)}$ is due to Witten's choice of a different normalization. The factors $H(\phi_i)$ result from our choice of a metric on the two projective planes.

Notice that there are two factors $H(\phi_i)$ (in contrast to the case when the manifold is a connected sum of a Riemann surface with one copy of P, when there is only one such factor). Thus $Vol(\mathcal{M})$ is a function of the angles ϕ_1, ϕ_2 for a fixed surface Σ_l.

We may summarize our conclusions as follows. When we take the connected sum of a Riemann surface with a real projective plane formed by identifying two sides of length L of a geodesic triangle with polar angle ϕ, the volume we have obtained is a function of the angle ϕ since the length L of the side of the triangle is determined once the length of the third edge of the triangle is given. A pair (L, ϕ) gives a conformal structure on the triangle. Once we fix a conformal structure, we obtain a volume for the moduli space determined by this structure. The situation is similar for the connected sum of a Riemann surface with two projective planes.

3. Witten's volume formula and Haar measure

In this section, we will consider Witten's volume formula for the moduli space of flat connections on a 2-manifold, and give a new proof of these formulas using Haar measure. Notice that Witten did not give a proof exclusively using Haar measure to compute the volume of the moduli space. Witten gave three arguments for the formula in [W]:

- In Section 2, Witten used results of Migdal [Mi] involving lattice gauge theories to compute the Yang-Mills partition function in two dimensions. Migdal's result only makes sense when the Yang-Mills partition function has been regularized, giving a result which reduces to the volume of the moduli space when the regularization parameter tends to 0.
- In Section 3, Witten deduced the result from the Verlinde formula, which is a formula for the Riemann-Roch number of the prequantum line bundle of the moduli space M of flat connections on a 2-manifold, $RR(M, L^k)$. Since the cohomology class of the symplectic form ω is the first Chern class of L, we may expect that the leading order term in the Riemann-Roch number (as a polynomial in k) is $k^{\dim M/2} Vol(M)$.
- In Section 4, Witten characterized the volume of the moduli space using Reidemeister-Ray-Singer torsion. He used the fact that Reidemeister torsion is multiplicative: If M is the union

of two submanifolds M_1 and M_2 with boundary N, glued along the boundary, then

$$\tau(M) = \tau(M_1)\tau(M_2)/\tau(N)$$

where the torsion is viewed as a ratio of determinants of elliptic operators and the above quotient makes sense in terms of the Mayer-Vietoris sequence which computes the cohomology groups of M from those of M_1, M_2 and N.

Since the moduli space can be identified with the space of representations of a surface group (in other words $\mathrm{Hom}(\pi_1(\Sigma), G)/G$) and a compact Lie group has a natural Riemannian measure, the Haar measure, we can try to understand Witten's formula for the moduli space by pushing forward the Haar measure on G.[3]

We will use the following two facts to show our argument.

- Weyl's integral formula (cf. [BD](4.1.11))

$$|W| \int_G f(g)dg = \int_T [F(t) \int_G f(gtg^{-1})dg]dt$$

If f is conjugation invariant, the formula becomes

$$
\begin{aligned}
|W| \int_G f(g)dg &= \int_T F(t)f(t)\frac{Vol(G)}{Vol(T)}dt \\
&= |W| \int_{T/W} i_*(F(t))f(i^{-1}(t))\frac{Vol(G)}{Vol(T)}i_*dt \\
&= |W| \int_{s \in T/W} F(i^{-1}(s))f(i^{-1}(s))\frac{Vol(G)}{Vol(T)}ds
\end{aligned}
$$

Here $i : T \to T/W$ is the local isomorphism with $i_*dt =: ds$.

Thus we have

$$\int_G f(g)dg = \int_{s \in T/W} F(s)f(s)\frac{Vol(G)}{Vol(T)}ds$$

Recall that $F(s) = F(i^{-1}(s))$ is the volume of the conjugacy class containing s as defined for example in [W](4.53):

$$v = \frac{Vol(G)}{Vol(T)}v_0 F(s),$$

where v represents the measure on T/W obtained by pushing down the Haar measure on G (under the natural map from a group element to its conjugacy class) and v_0 represents the measure on T/W obtained by restricting the metric on $\mathfrak{g}$ to $\mathrm{Lie}(T)$ (which determines a Haar measure

---

[3]For a related treatment of the role of Haar measure in determining volumes, see [AMM] and [AMW].

on T) and then identifying $\mathrm{Lie}(T)$ with the tangent space to T/W. This is as in the picture shown below:

$$
\begin{array}{ccc}
T & & G \\
& \searrow_{\;i} \quad \swarrow^{u}\; & \\
& T/W &
\end{array}
$$

with $v = u_* dg$, and $v_0 = i_* dt$, where dg and dt are the Haar measures on G and T respectively. Here $i : T \to T/W$ is the local isomorphism and $u : G \to T/W$ is the map which maps g to its conjugacy class.

Explicitly we have

$$
(46) \qquad\qquad F(s) = \det_{\mathbb{R}}(1 - \mathrm{Ad}(s))
$$

or for $s = \exp \lambda$ for $\lambda \in \mathrm{Lie}(T)$ such that $\alpha(\lambda) \neq 0$ for any root α, we have

$$
F(\exp \lambda) = \prod_{\alpha > 0} 4 \sin^2 \alpha(\lambda).
$$

where we assume $s \in T$ and view $\mathrm{Ad}(s)$ as an endomorphism of the orthocomplement of the Lie algebra of T.

• Pushforward of volume under a surjective map: Suppose $p : M \to N$ is a surjective map, $f : N \to R$ is a map, and V_M and V_N are volume forms for M and N respectively. Then a function h characterizing the pushforward is defined by

$$
\int_M f(p(x)) V_M = \int_N f(y) h(y) V_N, \ \text{ where } p_*(V_M)(y) = h(y) V_N.
$$

Roughly speaking, $h(y) = Vol(p^{-1}(y))$.

3.1. Σ is a one punctured Riemann surface of genus $\ell + 1$. The following picture shows the relations between the pushforwards.

$$[G \times G]^\ell \qquad G \times G$$

$$Q \searrow \swarrow R$$

$$G$$

$$\downarrow \pi$$

$$T/W$$

where $Q(A_1, \cdots, A_\ell, B_1, \cdots, B_\ell) = \prod_{i=1}^{\ell} A_i B_i A_i^{-1} B_i^{-1}$

(47) $\qquad$ and $R(g_1, g_2) = h g_1 g_2 g_1^{-1} g_2^{-1}$

where $h \in G$ is the holonomy around the boundary.

The volume formula for a moduli space of flat connections on a compact orientable surface with one boundary component [W](4.114) is as follows. For an element $s \in G$

(48) $\qquad Vol(\mathcal{R}(\Sigma_\ell, s)) = \dfrac{\sharp Z(G) Vol(G)^{2\ell-1}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \dfrac{1}{(\dim(\alpha))^{2\ell-1}} \chi_\alpha(s)$

We want to show that Witten's formula [W] (4.114) is the pushforward of the Haar measure on G i.e. prove equation (48) by induction on ℓ (assuming that equation (48) is true for genus ℓ, we prove it for genus $\ell + 1$). The induction hypothesis is valid for $\ell = 0$ because by [W] (2.43) we have

(49) $\qquad \sum_\alpha (\dim \alpha) \chi_\alpha(s) = \delta(s - 1)$

and the space $\mathcal{R}(\Sigma_0, s)$ is one point if $s = 1$ and empty otherwise, so its volume is $\delta(s - 1)$.

Consider a torus Σ_1 with one boundary component. If $\Sigma_1 - D$ is glued to Σ_ℓ along the boundary of D, we obtain $\Sigma_{\ell+1}$. (See Figure 3.)
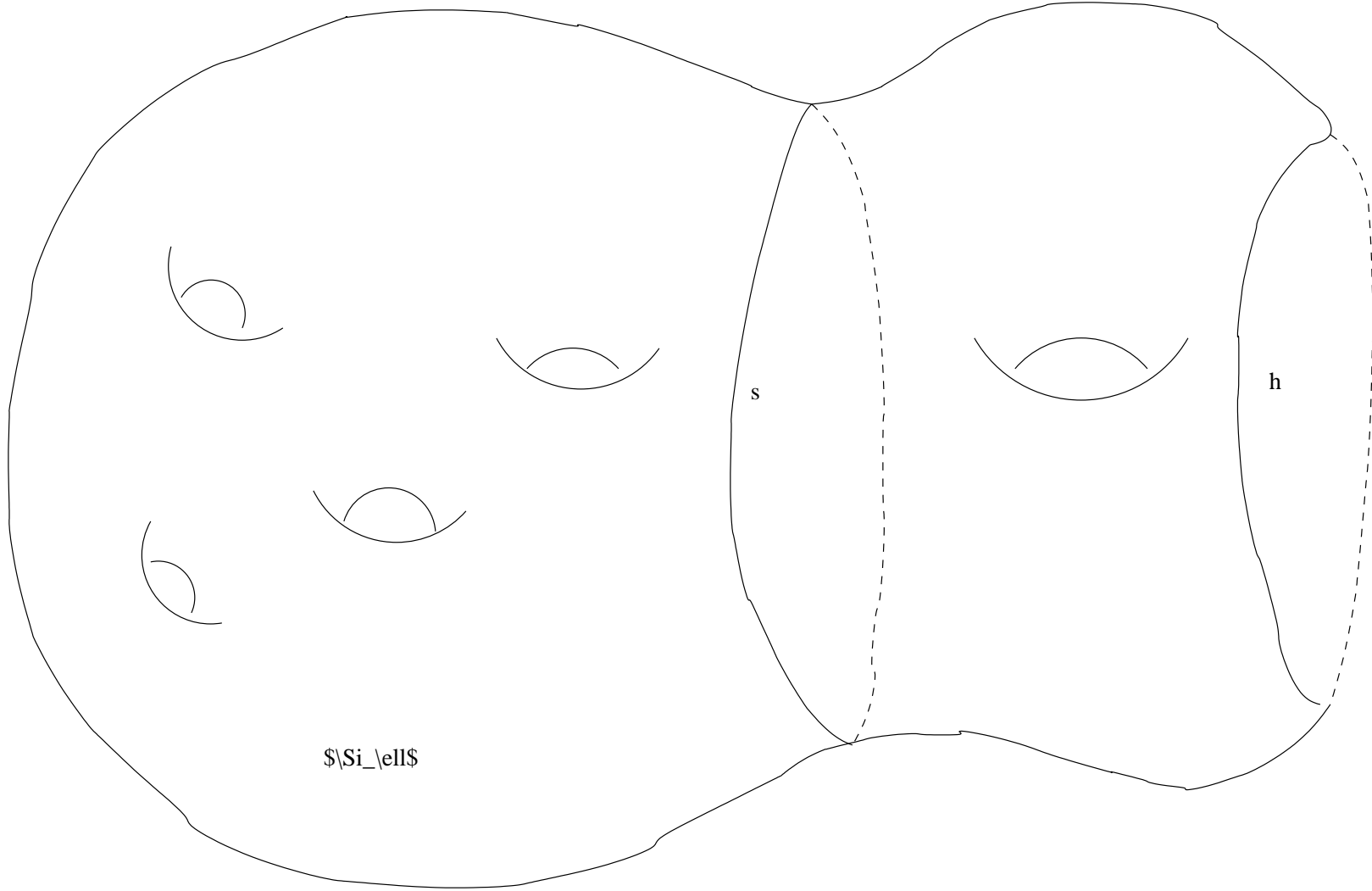
Figure 3: Connected sum of higher genus Riemann surface with punctured torus.

Let $h \in G$ be the holonomy around the boundary component of Σ_1 (which becomes the boundary component of $\Sigma_{\ell+1}$ after gluing to Σ_ℓ). We have

$$
\begin{aligned}
Vol(\mathcal{R}(\Sigma_{\ell+1}, h)) &= Vol(\mathcal{R}(\Sigma_\ell \sharp \Sigma_1, h)) \\
&= \int_{s \in G} Vol(\mathcal{R}(\Sigma_\ell, s)) Vol(\mathcal{R}(\Sigma_1, s, h)) ds
\end{aligned}
$$

Also, by definition of the pushforward we have

$$
(50) \qquad R_*(dg_1 \wedge dg_2) = Vol(\mathcal{R}(\Sigma_1, s, h)) ds.
$$

Thus we have

$$Vol(\mathcal{R}(\Sigma_{\ell+1}, h)) = \int_G Vol(\mathcal{R}(\Sigma_\ell, s)) Vol(\mathcal{R}(\Sigma_1, s, h)) ds$$

$$= \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \int_G \chi_\alpha(g) R_*(dg_1 \wedge dg_2)$$

$$= \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \int_{G \times G} \chi_\alpha(hg_1 g_2 g_1^{-1} g_2^{-1}) dg_1 dg_2$$

$$= \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \int_G \chi_\alpha(hg_1) \chi_\alpha(g_1^{-1}) dg_1 \frac{Vol(G)}{\dim(\alpha)}$$

where we have used[4] (cf:[W](2.50))

(51) $$\int_G \chi_\alpha(AuBu^{-1}) du = \frac{Vol(G)}{\dim(\alpha)} \chi_\alpha(A) \chi_\alpha(B)$$

and R was defined by (47).

Now we use (cf:[W](2.48))

(52) $$\int_G \chi_\alpha(hg) \chi_\alpha(g^{-1}) dg = \frac{Vol(G)}{\dim(\alpha)} \chi_\alpha(h)$$

to get our final formula for $h \in G$

(53) $$Vol(\mathcal{R}(\Sigma_{\ell+1}, h)) = \frac{\sharp Z(G) Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell+1)|G|-|T|}} \sum_\alpha \frac{\chi_\alpha(h)}{(\dim(\alpha))^{2\ell+1}}$$

We note that the final formula (53) is invariant under conjugation: $Vol(\mathcal{M}(\Sigma_{\ell+1}, h)) = Vol(\mathcal{M}(\Sigma_{\ell+1}, khk^{-1}))$ for any $k \in G$. So this formula only depends on the conjugacy class of h.

Note that Witten [W] (4.114) gives the following formula for the volume for genus $\ell + 1$ with $h \in G$,

$$Vol(\mathcal{R}(\Sigma_{\ell+1}, h)) = \frac{\sharp Z(G) Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell+1)|G|-|T|}} \sum_\alpha \frac{\chi_\alpha(h)}{(\dim(\alpha))^{2\ell+1}}$$

This agrees with our result.

Our calculation shows us that Witten's formula (4.114) can be understood in terms of the pushforward of Haar measure on G.

---

[4]The equations [W] (2.43), (2.48), (2.50), (2.70) and (4.50) are standard facts from the representation theory of compact Lie groups, and follow from the orthogonality relations. See also [BD].

3.2. Σ is the connected sum of a Riemann surface of genus ℓ with P.

We know from [W] the volume formula for a moduli space of flat connections on a compact orientable surface with one boundary component [W](4.114), where $s \in G$ and the holonomy around the boundary is constrained to be s:

$$Vol(\mathcal{R}(\Sigma_\ell, s)) = C_1 VolT \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \chi_\alpha(s)$$

where the constant C_1 is given by equation (36).

The following picture shows the relations between the pushforwards.

where

$$(54) \qquad Q(A_1, \cdots, A_\ell, B_1, \cdots, B_\ell) = \prod_{i=1}^{\ell} A_i B_i A_i^{-1} B_i^{-1}$$

and

$$(55) \qquad\qquad\qquad R(g) = g^2.$$

By definition of the pushforward,

$$(56) \qquad\qquad Vol(\mathcal{R}(P, g))dg = R_*(dg)$$

Thus

$$
\begin{aligned}
Vol(\mathcal{R}(\Sigma)) &= Vol(\mathcal{R}(\Sigma_\ell \sharp P)) = \int_{s \in G} Vol(\mathcal{R}(\Sigma_\ell, s)) Vol(\mathcal{R}(P, s)) ds \\
&= \int_G C_1 Vol T \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \chi_\alpha(g) R_*(dg) \\
&= C_1 Vol T \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \int_G \chi_\alpha(g^2) dg \\
&= \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} f_\alpha
\end{aligned}
$$

where we have used equation (42) and f_α is defined in Section 2.5.
   We get our final formula

$$
Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \frac{f_\alpha}{(\dim(\alpha))^{2\ell-1}}
$$

Note that Witten's formula [W](4.93) is

$$
Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|}} \sum_\alpha \frac{f_\alpha}{(\dim(\alpha))^{2\ell-1}}
$$

Our formula differs from Witten's by a multiplicative factor of $(2\pi)^{|T|}$ because Witten chose a different normalization.

3.3. Σ is the connected sum of a Riemann surface of genus ℓ with the Klein bottle. Again, the following picture shows the relations between the pushforwards. (Note that we will have a similar calculation if we use [W] (4.114) with two boundary components instead. It will be like the calculation in Section 2.6.)

where

$$Q(A_1, \cdots, A_\ell, B_1, \cdots, B_\ell) = \prod_{i=1}^{\ell} A_i B_i A_i^{-1} B_i^{-1}$$

and $R(g_1, g_2) = g_1 g_2 g_1^{-1} g_2$. We have that the volume of the moduli space of flat connections on a once punctured Klein bottle with holonomy around the puncture given by s is (by definition of the pushforward)

(57) $$Vol(\mathcal{R}(K, g))dg = R_*(dg_1 \wedge dg_2).$$

Moreover

$$Vol(\mathcal{R}(\Sigma)) = Vol(\mathcal{R}(\Sigma_\ell \sharp K)) = \int_{s \in G} ds Vol(\mathcal{R}(\Sigma_\ell, s)) Vol(\mathcal{R}(K, s))$$

Here we have used Lemma 6, which is again justified by [Go1] Proposition 3.7.

Thus we have

$$Vol(\mathcal{R}(\Sigma)) = \int_G C_1 VolT \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \chi_\alpha(g) R_*(dg_1 \wedge dg_2)$$

$$= C_1 VolT \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \int_{G \times G} \chi_\alpha(g_1 g_2 g_1 g_2^{-1}) dg_1 dg_2$$

$$= \frac{\sharp Z(G) Vol(G)^{2\ell}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_\alpha \frac{1}{(\dim(\alpha))^{2\ell-1}} \frac{1}{\dim(\alpha)} \int_G \chi_\alpha(g_1) \chi_\alpha(g_1) dg_1$$

where we have used equation (51). Thus

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_{\alpha = \bar{\alpha}} \frac{1}{(\dim(\alpha))^{2\ell}}$$

where we have used (cf:[W](4.50))

(58) $$\int_G \chi_\alpha(g) \chi_\alpha(g) dg = \begin{cases} 0 & \text{if } \alpha \neq \bar{\alpha} \\ Vol(G) & \text{if } \alpha = \bar{\alpha} \end{cases}$$

Thus our final formula is

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell-1)|G|-|T|}} \sum_{\alpha = \bar{\alpha}} \frac{1}{(\dim(\alpha))^{2\ell}}$$

We compare our result with Witten's formula [W](4.77):

$$Vol(\mathcal{R}(\Sigma)) = \frac{\sharp Z(G) Vol(G)^{2\ell+1}}{(2\pi)^{(2\ell)|G|}} \sum_{\alpha = \bar{\alpha}} \frac{1}{(\dim(\alpha))^{2\ell}}$$

Our formula differs from Witten's by a multiplicative factor of $(2\pi)^{|G|+|T|}$ because Witten chose a different normalization.

This gives us a better idea of the geometric meaning of Witten's volume [W]. It is the integral of the measure derived from the symplectic volume on the moduli space of flat connections on an orientable surface and the pushforward volume of the Haar measure on products of G. Since the symplectic volume of the orientable part can also be explained as the Haar measure of the Lie group model (as we did in section 3.1), this explains why the Haar measure on products of Lie groups gives Witten's formula for the volume on the moduli space of flat connections on a nonorientable 2-manifold.

Appendix A. Evaluation of an integral

In this appendix we compute the integral (23). We assume k is a positive integer, so

$$(59) \qquad \int_{\cos(\phi/2)}^{1} \left(\frac{x}{l} - \sqrt{(\frac{x}{l})^2 - 1} \ \right)^{2k} \frac{dx}{\sqrt{1 - x^2}} = S_1 + S_2,$$

where

$$(60) \quad S_1 = \sum_{s=1}^{k} \frac{(2k)!}{(2s)!(2k - 2s)!} \int_{\cos(\phi/2)}^{1} (\frac{x}{l})^{2s}((\frac{x}{l})^2 - 1)^{k-s} \frac{dx}{\sqrt{1 - x^2}}$$

and
$$(61)$$
$$S_2 = -\sum_{s=0}^{k-1} \frac{(2k)!}{(2s + 1)!(2k - 2s - 1)!} \int_{\cos\phi/2}^{1} (\frac{x}{l})^{2s+1} \left((\frac{x}{l})^2 - 1 \right)^{k-s-1} \sqrt{(\frac{x}{l})^2 - 1} \frac{dx}{\sqrt{1 - x^2}}.$$

We can use the trigonometric substitution $x = \cos\theta$ for the integral (60). This yields

$$\int_{\cos(\phi/2)}^{1} \sum_{s=1}^{k} \left(\frac{x^{2s}}{l^{2s}} \right) \left((\frac{x}{l})^2 - 1 \right)^{k-s} \frac{dx}{\sqrt{1 - x^2}}$$

$$= \int_{0}^{\phi/2} \sum_{s=1}^{k} \frac{(\cos^2\theta)^s}{l^{2s}} (\frac{\cos^2\theta}{l^2} - 1)^{k-s} d\theta$$

$$= \sum_{s=1}^{k} \sum_{r=0}^{k-s} \frac{1}{l^{2(s+r)}} (-1)^{k-s-r} \frac{(k - s)!}{r!(k - s - r)!} \int_{0}^{\phi/2} (\cos^2\theta)^{s+r} d\theta$$

$$= \sum_{s=1}^{k} \sum_{r=0}^{k-s} \frac{1}{l^{2(s+r)}} \frac{(k - s)!(-1)^{k-s-r}}{r!(k - s - r)!} \left[\frac{1}{2^{s+r}} \frac{(2s + 2r)!}{(s + r)!(s + r)!} \phi/2 \right.$$

$$+\frac{1}{2^{2(s+r)-1}}\sum_{p=0}^{s+r-1}\frac{(2s+2r)!}{p!(2s+2r-2p)!}\frac{\sin(s+r-p)\phi}{2(s+r-p)}\Bigg]$$

The integral (61) can be obtained by the substitution $y=\sqrt{1-x^2}$. Introducing $m^2=1-l^2$ and $y=mz$, this gives

$$\int_{\cos(\phi/2)}^{1}\sum_{s=0}^{k-1}\frac{(2k)!}{(2s+1)!(2k-2s-1)!}(\frac{x}{l})^{2s+1}((\frac{x}{l})^2-1)^{k-s-1}\sqrt{(\frac{x}{l})^2-1}\frac{dx}{\sqrt{1-x^2}}$$

$$=\sum_{s=0}^{k-1}\sum_{t=0}^{s}\int_{0}^{(1/m)\sin(\phi/2)}\frac{1}{l^{2k}}\frac{s!}{t!(s-t)!}(-m^2z^2)^t m^{2(k-s-1)}(1-z^2)^{k-s-1}m^2\sqrt{1-z^2}dz$$

Now let $z=\sin\theta$, and we obtain

$$\sum_{s=0}^{k-1}\sum_{t=0}^{s}\sum_{r=0}^{t}\frac{m^{2(k-s+t)}}{(1-m^2)^k}(-1)^{t+r}\frac{s!}{r!(t-r)!(s-t)!}\int_{0}^{\sin^{-1}[(1/m)\sin(\phi/2)]}(\cos^2\theta)^{k-s+r}d\theta$$

$$=\sum_{s=0}^{k-1}\sum_{t=0}^{s}\sum_{r=0}^{t}\frac{m^{2(k-(s-t))}}{(1-m^2)^k}(-1)^{t+r}\frac{s!}{r!(t-r)!(s-t)!}\times$$

$$\times\Bigg[\frac{1}{2^{2(k-(s-r))}}\frac{[2(k-s+r)]!}{(k-s+r)!(k-s+r)!}\theta_m$$

$$+\frac{1}{2^{2(k-s+r)-1}}\sum_{p=0}^{k-s+r-1}\frac{[2(k-s+r)]!}{p!(2[k-s+r]-p)!}\frac{\sin\{2\theta_m(k-s+r-p)\}}{2(k-s+r-p)}\Bigg]$$

where $\theta_m=\arcsin(1/m\sin(\phi/2))$.

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